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ON CASTELNUOVO'S INEQUALITY FOR ALGEBRAIC CURVES. I

BY

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ABSTRACT. Let W_p be a Riemann surface of genus p admitting a simple linear series g_n^r where $n = m(r-1) + q$, $q = 2, 3, \ldots, r-1$, or r. Castelnuovo's inequality states that (1) $2p \leqslant 2f(r, n, 1) = m(m-1)(r-1) + 2m(q-1)$. By further work of Castelnuovo, equality in (1) and $q < r$ implies that W_p admits a plane model of degree $n - r + 2$ with $r - 2$ m-fold singularities and one $(n - r + 1 - m)$-fold singularity. Formula (1) generalizes as follows. Suppose W_p admits s simple linear series g_n^r where $n = m(rs - 1) + q$ and $q = -(s-1)r + 2, -(s-1)r + 3, \ldots, r-1$, or r. For q consider the cases $v = 0, 1, \ldots, s-1$ as follows: case $v = 0$: $2 \leqslant q \leqslant r$, case $v > 0$: $2 \leqslant q + vr \leqslant r + 1$. Then (2) $2p \leqslant 2f(r, n, s) = m^2(rs^2 - s) + ms(2q - 1 - r) - v(v-1)r - 2v(q-1)$. Examples show that (2) is sharp. Finally, if $n = m'r + q'$, $q' = 1, 2, \ldots, r-1$, or r and W_p admits $m' + 1$ simple g_n^r's then (3) $2p \leqslant 2f(r+1, n+1, 1) = m'(m'-1)r + 2m'q'$. Since $f(r, n, 2) < f(r, n, 1)$ we obtain as a corollary: if $p = f(r, n, 1)$ then W_p admits at most one simple g_n^r.

1. Introduction. Let W_p be a closed Riemann surface of genus p admitting a simple linear seties g_n^r. Castelnuovo [2] showed that

$$p \leqslant (n - r + \varepsilon)(n - 1 - \varepsilon)/2(r-1), \tag{1.1}$$

where $0 \leqslant \varepsilon \leqslant r - 2$ and $n - r + \varepsilon \equiv 0 \pmod{r-1}$.

If $r = 2$ this is simply the fact that a plane curve of degree n has its genus bounded by $(n-1)(n-2)/2$. If W_p admits a simple g_n^2 where $p = (n-1)(n-2)/2$ then W_p admits a plane model as a nonsingular curve of degree n. Since the g_n^2 is unique the locus of all such W_p in Teichmüller space for genus p is $n(n+3)/2 - 8$ where $n(n+3)/2$ is the dimension of the (nonsingular) plane curves of degree n and 8 is the dimension of the plane collineations.

In this paper we will first generalize these classical results to arbitrary dimension r. By a beautiful theorem of Castelnuovo [1] equality in formula (1.1) will insure that W_p admits a certain type of plane model. Moreover g_n^r will be unique in this case. By classical dimension counting we can then

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derive the dimension in Teichmüller space of the Riemann surfaces W_p admitting such g_n^r's. Also we will be able to say something about the automorphism groups of such surfaces.

Let us put the problem in a more general context. Suppose W_p admits s linear series, $g_{n_1}^{r_1}, g_{n_2}^{r_2}, \ldots, g_{n_s}^{r_s}$ which are simple if $r_j \geqslant 2$ (and perhaps other suitable hypotheses). Then there is an integer valued function $f(r_1, r_2, \ldots, r_s; n_1, n_2, \ldots, n_s)$ such that

$$p \leqslant f(r_1, \ldots, r_s; n_1, \ldots, n_s). \tag{1.2}$$

The problem is to determine the function f and investigate the consequences for W_p of equality in (1.2). If $s = 1$, formula (1.2) is Castelnuovo's inequality, formula (1.1).

We shall derive the function f in the case of s ($s \geqslant 2$) simple g_n^r's on W_p all of the same dimension and degree. Plane curves will be exhibited to show that the derived function f is the best possible. It will be observed that $f(r, r; n, n) < f(r; n)$, so that if $f(r, r; n, n) < p \leqslant f(r; n)$ then a simple g_n^r on W_p must be unique.

In Part II of this paper we will consider the consequences of equality in formula (1.2) in the case of W_p admitting several simple g_n^r's of the same degree and dimension. The problem will be to show that the plane models exhibited in §7 of this paper are the only possible ones, at least for large dimension r.

2. Notation, definitions, and preliminary results. A closed Riemann surface of genus p will be denoted W_p. We will always assume that W_p is not hyperelliptic. A linear series of dimension r and degree n will be denoted g_n^r. Such a series may have fixed points, may be simple or composite, and may be complete or incomplete. For x in W_p, $g_n^r - x$ will denote the linear series of degree $n - 1$ of divisors in g_n^r passing through x, not counting x. If x is not a fixed point of g_n^r then $g_n^r - x = g_{n-1}^{r-1}$. If g_n^r is simple and without fixed points then for a general choice of x, $g_n^r - x$ will also be simple and without fixed points.

If g_n^r is composite then W_p is a t-sheeted covering of a Riemann surface of genus q, W_q, and a divisor of nonfixed points of g_n^r is a union of fibers of the map from W_p onto W_q. The set of fibers will be called an *involution* following classical terminology, and will be denoted γ_t. In this case we will say that g_n^r is *compounded* of the involution γ_t. Note that a given g_n^r may be compounded of several different involutions. We will say that a g_n^1 is compounded of an involution if each divisor in g_n^1 is made up of divisors in the involution. If g_n^1 is compounded only of itself and the trivial involution then we will say that g_n^1 is simple; otherwise, g_n^1 will be called composite.

The Riemann-Roch theorem states that for a complete g_n^r, $r = n - p + i$

where i is the index of speciality. Let $K = g_{2p-2}^{p-1}$ denote the canonical series which we are assuming is always simple. If $g_n^r + g_{n'}^{r'} = K$ where both series are complete, then the Brill-Noether formulation of the Riemann-Roch theorem states that $n - 2r = n' - 2r'$. For special g_n^r's Clifford's theorem states that $n - 2r \geqslant 0$. For an arbitrary g_n^r the integer $n - 2r$ will be called the *Clifford index*. If the Clifford index of a series is negative then the series is not special.

As usual the greatest common divisor of the two divisors D and E will be denoted (D, E).

If g_n^r is a complete linear seties, a second series g_m^s is said to impose t (linear) conditions on g_n^r if there is a complete series g_{n-m}^{r-t} so that $g_n^r = g_m^s + g_{n-m}^{r-t}$. This means that if D is any divisor in g_m^s of m distinct points, then there are t points of D, $x_1, \ldots, x_t$ so that $g_n^r - (x_1 + \cdots + x_t)$ has the other points of D among its fixed points. Also $x_1, \ldots, x_t$ impose independent conditions; that is, for each k there is a divisor in g_n^r containing all the x's except x_k. If g_m^1 imposes one condition on g_n^r then $g_n^r = rg_m^1 + D_{n-rm}$ where D_{n-rm} is the divisor of fixed points of the composite g_n^r. We shall often use the fact that a g_m^1 without fixed points imposes $m - 1$ conditions on the canonical series.

We will say that a linear series g_m^s is included in g_n^r if for any E in g_m^s there is a D in g_n^r so that $(E, D) = E$. If g_n^r is complete this means that g_m^s imposes r or fewer conditions on g_n^r. To say that g_m^s imposes $r + 1$ conditions on a complete g_n^r will mean that g_m^s is not included in g_n^r.

If g_n^r is simple, $r \geqslant 2$, and without fixed points then W_p can be realized as a curve in P^r and the hyperplane sections cut out the divisors of g_n^r on the curve. To say that g_m^s imposes t conditions on a simple g_n^r means geometrically that every divisor in g_m^s spans a linear space of dimension $t - 1$ in P^r. In general we will say that t points in P^r are independent if they span a linear space of dimension $t - 1$. The points of a divisor D in g_n^r will be said to be in general position if any r of them are independent and so span the hyperplane which cuts out D on the curve in P^r.

In the case where g_n^2 is simple and without fixed points, W_p admits a plane model of degree n. If d is the number of double points suitably counted, then

$$p = (n - 1)(n - 2)/2 - d. \tag{2.1}$$

To compute the dimension R of all plane curves of degree n with s given ordinary singularities of multiplicities $k_1, k_2, \ldots, k_s$ we use the classical formula

$$R \geqslant n(n + 3)/2 - \sum_{j=1}^{s} k_j(k_j + 1)/2. \tag{2.2}$$

If $s \leqslant 3$ this formula is precise, as it will be in the applications in Part I of this paper.

A singularity of multiplicity k will be called a *k-fold point* of a curve or linear series. Thus if $x_1 + \cdots + x_k$ is a k-fold point for the simple g_n^r, then $g_n^r - x_j$ has the other $k - 1$ x's as the divisor of fixed points. g_n^r will be called nonsingular if $g_n^r - x$ has no fixed points for all x in W_p. If Q_1 and Q_2 are singularities for g_n^r we will say that they are disjoint as point sets on W_p if $(Q_1, Q_2) = 1$. This does not exclude the possibility that the two singularities may occur at the same point in P^r on the curve given by the mapping of W_p into P^r associated with g_n^r.

Let $(W_p)^t$ denote the t-fold symmetric product of W_p; that is, the space of all integral divisors of degree t.

If g_n^r is a simple linear series we shall have occasion to speak of the "general divisor" of g_n^r having a certain property. By this we shall mean the following. There is a dense open set O in $(W_p)^r$ so that if $x_1 + \cdots + x_r$ is in O then these points uniquely determine a divisor D in g_n^r (they impose independent conditions) and D has that property.

Finally we shall discuss a method of Castelnuovo which is basic. Let $g_{n_1}^{r_1}, \ldots, g_{n_k}^{r_k}$ be k linear series. Let $g_N^R = g_{n_2}^{r_2} + \cdots + g_{n_k}^{r_k}$ and let $g_{N_1}^{R_1} = g_N^R + g_{n_1}^{r_1}$. Castelnuovo's method allows one to obtain a lower bound on R_1. Suppose that $g_{n_1}^{r_1}$ imposes $t_j + 1$ conditions on $g_{n_j}^{r_j}$ for all j. Clearly $g_{n_1}^{r_1}$ imposes r_1 conditions on $g_{n_1}^{r_1}$. If $r_j \geqslant r_1$ then typically in this paper $t_j + 1 \geqslant r_1 + 1$. If $r_j \leqslant r_1$ then typically in this paper $t_j + 1 \geqslant r_j + 1$. Then we can find a divisor D in $g_{n_1}^{r_1}$ and divisors E_j in $g_{n_j}^{r_j}$ so that (D, E_j) has order t_j for all j. Also the totality of $t_1 + \cdots + t_k$ points in these k divisors all impose independent conditions on $g_{N_1}^{R_1}$. Consequently we see that if $g_{n_1}^{r_1}$ imposes T conditions on $g_{N_1}^{R_1}$ then $T \geqslant t_1 + \cdots + t_k + 1$. Since $R = R_1 - T$ we have $R_1 \geqslant R + t_1 + \cdots + t_k + 1$. This is the estimate we want. It is usually the kth step in an inductive argument. That we can find such divisors E_j follows from theorems about the general position of points on divisors in linear series on curves in projective space. One of these theorems on general position, Theorem 3.1, is classical and the other, Theorem 4.1, follows easily from known results. (This latter theorem, no doubt, was known classically but the author knows of no reference.)

3. Castelnuovo's method for one linear series. Most of the results of this section are due to Castelnuovo, [1], [2]. The following basic theorem seems to be part of the folklore of the subject.

3.1 THEOREM. *Let g_n^r be a simple linear series without fixed points. Then the general divisor D of g_n^r satisfies the following*: (1) *D is made up of n distinct points, and* (2) *any r points of D impose independent conditions on g_n^r.*

PROOF. See [4, pp. 266–267, Theorems 39 and 43].

DEFINITION. For r and l nonnegative integers let $R(l; r) = l(l+1)r/2 - l(l-1)/2$.

Note that $R(l+1, r) = R(l; r) + (l+1)(r-1) + 1$.

3.2 LEMMA (CASTELNUOVO). *Let g_n^r be a simple linear series on W_p. If $k(r-1) + 1 \leqslant n$ then $kg_n^r = g_{kn}^{R(k;r)+\varepsilon}$ where $\varepsilon \geqslant 0$.*

PROOF. We use induction on k. We ask how many linear conditions g_n^r imposes on kg_n^r. If $D_1, D_2, \ldots, D_k$ are divisors in g_n^r then $D_1 + D_2 + \cdots + D_k$ is a divisor in kg_n^r. Let D_0 be a divisor in g_n^r whose n points are in general position. Then we can find divisors $D_1, \ldots, D_k$ in g_n^r so that for each j (D_0, D_j) is a divisor of degree $r - 1$. This follows from Theorem 3.1. Thus g_n^r imposes at least $k(r-1) + 1$ conditions on kg_n^r. But $kg_n^r - g_n^r = (k-1)g_n^r = g_{(k-1)n}^{R(k-1;r)+\varepsilon}$. Thus the dimension of kg_n^r is at least $R(k-1; r) + k(r-1) + 1$ $(= R(k; r))$. Q.E.D.

We now prove Castelnuovo's inequality, formula (1.1) in a slightly different form.

3.3 THEOREM (CASTELNUOVO). *Let W_p admit a simple g_n^r $(r \geqslant 2)$. Write $n = m(r-1) + q$ where $q = 2, 3, \ldots, r-1$, or r. Then*

$$p \leqslant m(m-1)(r-1)/2 + m(q-1). \tag{3.1}$$

PROOF. By Lemma 3.2, $mg_n^r = g_{mn}^{R(m;r)+\varepsilon}$. The Clifford index of this series is $mn - 2R(m; r) - 2\varepsilon$ which is seen to be negative. Thus the series is nonspecial and so $p \leqslant mn - R(m; r)$. This is formula (3.1). Q.E.D.

We are interested in the case where we have equality in Castelnuovo's inequality. In this case we see that the dimension of mg_n^r is precisely $R(m; r)$. By the proof of Lemma 3.2 we see that this implies that the dimension of kg_n^r is precisely $R(k; r)$ for all k less than or equal to m. From now on let us assume that $m \geqslant 2$ and if $m = 2$ then $q \neq 2$, putting aside the uninteresting cases $m = 1$, and $m = 2$, $q = 2$, the canonical series. Consequently n is always greater than $2r$. If now $k = 2$ we have $2g_n^r = g_{2n}^{3r-1}$. In this situation Castelnuovo [1] proved a beautiful theorem which says that the curve C in P^r, given by the simple series g_n^r, lies on an algebraic surface of degree $r - 1$. If $r \neq 5$ this means that the surface is a rational normal scroll whose rulings cut out on C a g_T^1 which imposes two conditions on g_n^r. If $r = 5$ then the algebraic surface could be the Veronese variety. In this latter case our curve C is the image of a plane curve C' and the conics cut out g_n^r on C'. We summarize these results in the following theorem.

3.4 THEOREM (CASTELNUOVO). *Suppose g_n^r is a simple linear series without fixed points so that $n > 2r$ and $2g_n^r = g_{2n}^{3r-1}$ which is complete. If $r \neq 5$ then W_p admits a g_T^1 which imposes two conditions on g_n^r. If $r = 5$ it may happen that W_p admits a plane model C' where the conics cut out g_n^r on C'.*

In his discussion of this theorem Castelnuovo stated that $T = m + 1$ or $m + 2$, the latter case occuring only if $q = r$. He did not appear to give a proof of this assertion. We will include a proof of this fact later.

3.5 LEMMA. *Suppose we have equality in Castelnuovo's inequality, formula* (3.1). *Then there is a complete linear series $g_{(q-2)(m+1)}^{q-2}$ so that*

$$K \equiv (m-1)\, g_n^r + g_{(q-2)(m+1)}^{q-2}. \tag{3.2}$$

Also

$$g_n^r + g_{(q-2)(m+1)}^{q-2} \equiv g_{n+(q-2)(m+1)}^{r+2(q-2)}. \tag{3.3}$$

PROOF. Since we have equality in formula (3.1) it follows that $(m-1)\, g_n^r = g_{(m-1)n}^{R(m-1;r)}$. This series is complete and special. The Clifford index is seen to be $(q-2)(m-1)$. Since $2p - 2 - (m-1)n = (q-2)(m+1)$ the result follows by the Brill-Noether form of the Riemann-Roch theorem. Formula (3.3) is proven in the same way using the fact that $g_n^r + g_{(q-2)(m+1)}^{q-2}$ has the same Clifford index as $(m-2)\, g_n^r$, namely $(m-2)(q+r-3)$. Q.E.D.

We now use Lemma 3.5 to get information about the degree of the g_T^1 which we know to exist on W_p.

3.6 LEMMA. *Suppose we have equality in formula* (3.1). *Suppose W_p admits a g_T^1. Then $T \geqslant m + 1$.*

PROOF. Suppose $T \leqslant m$. Let E $(= x_1 + x_2 + \cdots + x_T)$ be a divisor of distinct points in g_T^1 which does not contain any fixed points of $g_{(q-2)(m+1)}^{q-2}$. Since g_n^r is simple, g_T^1 must impose at least two conditions on g_n^r. Therefore, we can assume that there are $T - 1$ divisors $D_1, D_2, \ldots, D_{T-1}$ in g_n^r so that $(E, D_j) = x_j$ for $j = 1, 2, \ldots, T-1$. Since g_T^1 imposes $T - 1$ conditions on K, we see that x_T must be in $D_1 + D_2 + \cdots + D_{T-1}$, a contradiction. Q.E.D.

3.7 LEMMA. *Suppose we have equality in formula* (3.1) *and suppose W_p admits a g_T^1 where $T = m + 1$. Then g_T^1 imposes precisely two conditions on g_n^r. Also $g_{(q-2)(m+1)}^{q-2} = (q-2)\, g_T^1$.*

PROOF. As in the preceding proof let E $(= x_1 + x_2 + \cdots + x_{m+1})$ be a general divisor in g_{m+1}^1 and choose $D_1, D_2, \ldots, D_{m-2}$ to be divisors in g_n^r so that $(D_j, E) = x_j$ for $j = 1, 2, \ldots, m-2$. Now let D_{m-1} contain $x_{m-1} + x_m$. $D_1 + D_2 + \cdots + D_{m-1}$ contains $x_1 + x_2 + \cdots + x_m + x_{m+1}$ and therefore x_{m+1} is in D_{m-1}. But the numbering is irrelevent; that is, whenever D_{m-1} contains $x_m + x_{m-1}$ it contains all the x's.

If we choose $D_1, D_2, \ldots, D_{m-1}$ so that $(D_j, E) = x_j$ for $j = 1, 2, \ldots, m - 1$, then whenever $g^{q-2}_{(q-2)(m+1)}$ contains x_m it also contains x_{m+1}. Again the numbering is irrelevent. Thus g^1_{m+1} imposes one condition on $g^{q-2}_{(q-2)(m+1)}$. Q.E.D.

4. Castelnuovo's method for several linear series. We now generalize the results of the last section.

4.1 THEOREM. *Suppose g^r_n and g^s_m are two different linear series without fixed points so that $r \geqslant s$. If both series are composite suppose that there is no involution of which both are compounded. Then for the general divisor D in g^r_n and any divisor E in g^s_m,* order$(D, E) \leqslant s$.

PROOF. *Step* (i). Suppose $r = s$. We assert that there is a divisor D in g^r_n so that if E is any divisor in g^r_m then order$(D, E) \leqslant r$.

Suppose not. Then for all divisors D in g^r_n there is a divisor E in g^r_m so that order$(D, E) \geqslant r + 1$. In $(W_p)^r$ fix a point $X^0 = x^0_1 + x^0_2 + \cdots + x^0_r$ so that for $X = x_1 + x_2 + \cdots + x_r$ in a neighborhood N of X^0 the following is true: $x_1, x_2, \ldots, x_r$ impose independent conditions on g^r_n (respectively, g^r_m) and determine a divisor D of n (respectively, E of m) distinct points. We can also assume that D (respectively, E) contains a point x_{r+1} so that $x_1, \ldots, x_{r+1}$ lie in different divisors of any involution of which g^r_n (respectively, g^r_m) is compounded. Now let $g^1_{n'}$ (respectively, $g^1_{m'}$) be the nonfixed points of the complete linear series $g^r_n - (x_1 + \cdots + x_{r-1})$ (respectively, $g^r_m - (x_1 + \cdots + x_{r-1})$). As x_r varies it determines a divisor D' in $g^1_{n'}$ (respectively, E' in $g^1_{m'}$) which contains x_{r+1}. Thus $g^1_{n'}$ and $g^1_{m'}$ are compounded of an involution a divisor of which contains the pair $x_r + x_{r+1}$. It follows that for any $x_1 + x_2 + \cdots + x_r$ in N the corresponding D and E contain $x_r + x_{r+1}$ lying in some involution of W_p. The order of these involutions is bounded by n and so W_p admits only a finite number of such involutions [3]. This implies that there is an involution common to g^r_n and g^r_m. This contradiction proves step (i).

Step (ii). Assume $s < r$. We assert that there is a divisor D in g^r_n so that if E is any divisor in g^s_m then order$(D, E) \leqslant s$. This follows from step (i) by taking a g^s_n in g^r_n so that there is no involution common to those of which g^s_n and g^s_m are compounded.

Step (iii). The proof of the theorem now follows since the conditions on $(W_p)^r$ that (D, E) have order $\geqslant s + 1$ are analytic. Q.E.D.

REMARKS. It follows that any s points of the general D impose independent conditions on g^s_m. Also it follows that either g^s_m is not included in g^r_n or g^s_m imposes at least $s + 1$ conditions on g^r_n. The condition that g^r_n and g^s_m have no involution of which each is compounded is equivalent to the condition that the two fields of meromorphic functions determined by g_n^r and g_m^s generate the full field on W_p.

DEFINITION. Suppose $l_1, l_2, \ldots, l_s; r_1, r_2, \ldots, r_s$ are nonnegative integers so that $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_s \geqslant 1$. Let

$$R(l_1, \ldots, l_s; r_1, \ldots, r_s) = \sum_{j=1}^{s} R(l_j; r_j) + \sum_{i<j} l_i l_j r_j.$$

Note that

$$R(l_1, \ldots, l_k + 1, \ldots, l_s; r_1, \ldots, r_s) = R(l_1, \ldots, l_k, \ldots, l_s; r_1, \ldots, r_s) + (r_k - 1)(l_k + 1) + 1 + r_k \sum_{i=1}^{k-1} l_i + \sum_{j=k+1}^{s} l_j r_j.$$

4.2 LEMMA. *Suppose W_p admits s linear series $g_{n_1}^{r_1}, \ldots, g_{n_s}^{r_s}$ where $r_1 \geqslant \cdots \geqslant r_s \geqslant 1$. Suppose that any linear series of dimension two or more is simple and that any two series of dimension one are not compounded of the same involution. Suppose that for $j = 1, 2, \ldots, s$ the nonnegative integers l_j satisfy $\Sigma_{i=j}^{s} l_i r_i + 1 - l_j \leqslant n_j$. Then*

$$\sum_{j=1}^{s} l_j g_{n_j}^{r_j} = g^{R(l_1, \ldots, l_s; r_1, \ldots, r_s) + \varepsilon}_{\Sigma l_j n_j}$$

where $\varepsilon \geqslant 0$.

PROOF. First we consider induction on s. If $s = 1$, this is Castelnuovo's method since $R(l - 1; r) + l(r - 1) + 1 = R(l; r)$ provided $l(r - 1) + 1 < n$. Now assuming it is true for some $s \geqslant 1$ we use induction on l_1. It is true for $l_1 = 0$. Suppose the lemma is true for $(l_1 - 1)g_{n_1}^{r_1} + \Sigma_{j=2}^{s} l_j g_{n_j}^{r_j}$. Then $g_{n_1}^{r_1}$ imposes at least N $(= l_1(r_1 - 1) + \Sigma_{j=2}^{s} l_j r_j + 1)$ conditions on $\Sigma_{j=1}^{s} l_j g_{n_j}^{r_j}$ provided $N \leqslant n_1$. For by Theorem 4.1, $g_{n_j}^{r_j}$ imposes at least $r_j + 1$ conditions of $g_{n_1}^{r_1}$ $(j \geqslant 2)$ and $g_{n_1}^{r_1}$ imposes r_1 conditions on $g_{n_1}^{r_1}$. Since

$$R(l_1 - 1, l_2, \ldots, l_s; r_1, r_2, \ldots, r_s) + l_1(r_1 - 1) + \sum_{j=2}^{s} l_j r_j + 1 = R(l_1, \ldots, l_s; r_1, \ldots, r_s)$$

the lemma follows. Q.E.D.

4.3 THEOREM (A GENERALIZED CASTELNUOVO INEQUALITY). *Let W_p admit s different linear series g_n^r, all of the same dimension and degree, $s \geqslant 2$. If $r \geqslant 2$ assume all the series are simple and if $r = 1$ assume that no two of the series are compounded of the same involution. Let $n = m(rs - 1) + q$ where q is the residue of n modulo $(rs - 1)$ so that $-(s - 1)r + 2 \leqslant q \leqslant r$. Divide the possibilities for q into s cases indexed by $k = 1, 2, \ldots, s$ as follows:*

$$
\begin{array}{ll}
k = 1, & -r(s-1)+2 \leq q \leq -r(s-2)+1, \\
k = 2, & -r(s-2)+2 \leq q \leq -r(s-3)+1, \\
\vdots & \vdots \\
k = k, & -r(s-k)+2 \leq q \leq -r(s-k-1)+1, \\
\vdots & \vdots \\
k = s-1, & -r+2 \leq q \leq 1, \\
k = s, & 2 \leq q \leq r.
\end{array}
$$

Let $v = s - k$. *Then*

$$2p \leq m^2(rs^2 - s) + ms(2q - 1 - r) - v(v-1)r - 2v(q-1). \quad (4.1)$$

PROOF. Let us order the s linear series and denote them $g_1, g_2, \ldots, g_s$. Then by Lemma 4.2 the Clifford index of $m(g_1 + g_2 + \cdots + g_k) + (m-1)(g_{k+1} + \cdots + g_s)$ $(= g_{(ms-v)n}^{R(m,\ldots,m,m-1,\ldots,m-1;r,\ldots,r)+\varepsilon})$ is negative, and so the series is not special. Therefore

$$p \leq (ms - v)n - R(m, \ldots, m, m-1, \ldots, m-1; r, \ldots, r).$$

Now compute to get the result.[3] Q.E.D.

Suppose we have equality in formula (4.1) Then by the method of Lemma 4.2 we see that if

$$l_j \leq m \quad \text{for } j = 1, \ldots, k; \qquad l_j \leq m-1 \quad \text{for } j = k+1, \ldots, s, \quad (4.2)$$

then $\sum_{j=1}^{s} l_j g_j = g_{n(l_1+\cdots+l_s)}^{R(l_1,l_2,\ldots,l_s;r,r,\ldots,r)}$, where the latter series is complete. Unless we have equality in all s cases of formula (4.2), this series is seen to be special and the Clifford index can then be computed. This method yields the following lemma.

4.4 LEMMA. *Suppose we have equality in formula* (4.1). *Then there is a complete linear series* $g_{(q-2+vr)(ms-v+1)+m(k-1)}^{q-2+vr}$ *so that*

$$K = m(g_1 + \cdots + g_{k-1}) + (m-1)(g_k + \cdots + g_s) + g_{(q-2+vr)(ms-v+1)+m(k-1)}^{q-2+vr}. \quad (4.3)$$

Now let $T = ms - v + 1$. Notice that there are $T - 2$ $(= m(k-1) + (m-1)(s-k+1))$ of the g_j on the left-hand side of formula (4.3). Just as Lemma 3.6 and 3.7 follow from Lemma 3.5, so the following lemmas follow from Lemma 4.4.

[3]The author apologizes for the computations involved. For future reference we make the following definitions. Let $R^{(l)} = R(m, \ldots, m, m-1, \ldots, m-1; r, \ldots, r)$ where there are l m's and $s - l$ $(m-1)$'s. Let $N^{(l)} = ((m-1)s + l)n$ and let $C^{(l)} = N^{(l)} - 2R^{(l)}$. Then some tedius computations confirm that $2R^{(l)} = m^2(rs^2 - s) + m(rs + s - 2(s-l)(rs-1)) + (s-l)((s-l-1)r - 2)$ and $C^{(l)} = (s(m-1) + l)(q + r(s-l-1) - 2) + lm$. For $l = k$ $C^{(k)}$ is negative when one remembers the restrictions on q. Then $p \leq C^{(k)} + R^{(k)}$.

4.5 LEMMA. *Suppose we have equality in formula* (4.1). *Suppose W_p admits a linear series g_t^1 without fixed points. Then $t \geqslant T$.*

PROOF. Suppose $t \leqslant T - 1$. Then we can find a divisor $x_1 + \cdots + x_t$ in g_t^1 so that $x_1, \ldots, x_{t-1}$ are in separate divisors of $m(g_1 + \cdots + g_{k-1}) + (m - 1)(g_k + \cdots + g_s)$ and x_t occurs nowhere. This is a contradiction. Q.E.D.

4.6 LEMMA. *Suppose we have equality in formula* (4.1) *and W_p admits a g_T^1. Then g_T^1 imposes two conditions on each g_j, $j = 1,2, \ldots, s$, and one condition on $g_{(q-2+vr)T+m(k-1)}^{q-2+vr}$.*

PROOF. This follows from formula (4.3) as Lemma 3.7 followed from formula (3.2).

5. Extensions of Castelnuovo's method. The following can be viewed as a generalization of Castelnuovo's theorem (Theorem 3.4). We include some results that will be used in Part II of this paper.

5.1 LEMMA. *Suppose g_n^r is simple and g_m^s is a different linear series, possibly composite, so that $r \geqslant s$. Then $g_n^r + g_m^s = g_{n+m}^{r+2s+\varepsilon}$ where $\varepsilon \geqslant 0$.*

PROOF. By Theorem 4.1 it follows that g_m^s imposes at least $s + 1$ conditions on g_n^r. Thus we can find a divisor D_0 in g_n^r and divisors D and E in g_n^r and g_m^s, respectively, so that the order of (D, D_0) is $r - 1$ and the order of (E, D_0) is s. It follows that g_n^r imposes at least $r + s$ conditions on $g_n^r + g_m^s$. The result follows. Q.E.D.

5.2 LEMMA. *Suppose g_n^r is simple and g_m^s is another series so that both are without fixed points and $r > s$. Suppose that the dimension of $g_n^r + g_m^s$ is precisely $r + 2s$. Finally suppose that g_m^s is simple, $s \geqslant 2$. Then there are precisely three possibilities for s: (1) $s = 2$, $r = 5$, $g_n^5 = 2g_m^2$ and $n = 2m$; (2) $s = r - 2$ and $g_n^r = g_m^{r-2} + g_{n-m}^1$; (3) $s = r - 1$ and $g_n^r = g_m^{r-1} + g_{n-m}^0$.*

PROOF. It follows from the proof of Lemma 5.1 that g_m^s imposes precisely $s + 1$ conditions on g_n^r and since $s + 1 \leqslant r$ we have $g_n^r = g_m^s + g_{n-m}^{r-s-1}$.

Case (1): Assume $g_m^s = g_{n-m}^{r-s-1}$. Then $r = 2s + 1$ and $g_n^{2s+1} = g_m^s + g_{n-m}^s = g_n^{3s-1+\varepsilon}$ where $\varepsilon \geqslant 0$. Consequently $s = 2 - \varepsilon$. Since $s \geqslant 2$, $\varepsilon = 0$ and $r = 5$. Case (2): $g_m^s \neq g_{n-m}^{r-s-1}$ and $s \geqslant r - s - 1$. Then by Castelnuovo's method (g_m^s imposes at least $(s - 1) + (r - s - 1) + 1$ conditions on $g_m^s + g_{n-m}^{r-s-1}$) we see that $g_n^r = g_n^s + g_{n-m}^{r-s-1} = g_n^{s+2r-2s-2+\varepsilon}$, or $s = r - 2 + \varepsilon$. If $s = r - 2$ then $g_n^r = g_m^{r-2} + g_{n-m}^1$. If $s = r - 1$ then $g_n^r = g_m^{r-1} + g_{n-m}^0$. Case (3): $s < r - s - 1$. Then again by Castelnuovo's method we see that g_m^s imposes at least $2s$ conditions on $g_m^s + g_{n-m}^{r-s-1}$. Thus $g_n^r = g_n^{2s+r-s-1+\varepsilon}$, or $s = 1 - \varepsilon$. This is impossible. Q.E.D.

We see then that if g_n^r is simple, $r > s$, $g_n^r + g_m^s$ has dimension $r + 2s$, $r \neq 5$, and $2 \leqslant s \leqslant r - 3$ then g_m^s is composite. (Thus the method of Lemma 4.2 cannot give sharp results when applied to linear series where the r_j are quite different.) We now explore the consequences of this situation.

5.3 Lemma. *Suppose g_n^r is simple and without fixed points, g_m^s is composite and without fixed points, and the dimension of $g_n^r + g_m^s$ is precisely $r + 2s$. Then there is a complete g_t^1, $st = m$, so that $g_m^s = sg_t^1$. Moreover, g_t^1 imposes precisely two linear conditions on g_n^r.*

Proof. Let g_m^s be compounded of γ_t. Suppose the general divisor of γ_t imposes b conditions on the simple g_{n+m}^{r+2s} $(= g_n^r + g_m^s)$. Then $b \geqslant 2$. If $u \leqslant s - 1$ then $g_{n+m}^{r+2s} - u\gamma_t$ is simple since it includes g_n^r. Consequently if $u \leqslant s - 1$ then the general divisor of γ_t imposes at least two conditions on $g_{n+m}^{r+2s} - u\gamma_t$. Thus

$$g_{n+m}^{r+2s} - (s-1)\gamma_t = g_n^r + g_{m-(s-1)t}^1$$
$$= g_{n+m-(s-1)t}^{r+2+c} = g_{n+m-(s-1)t}^{r+2s-b_1-b_2-\cdots-b_{s-1}}$$

where $c \geqslant 0$ and $b_j \geqslant 2$ for $j = 1,2, \ldots, s-1$. Thus

$$r + 2 + c = r + 2s - b_1 - b_2 - \cdots - b_{s-1} \leqslant r + 2s - 2(s-1),$$

or $2 + c \leqslant 2$. Therefore $c = 0$ and we see that $g_{m-(s-1)t}^1$ imposes two conditions on $g_{n+m-(s-1)t}^{r+2}$. Since γ_t also imposes two conditions on $g_{n+m-(s-1)t}^{r+2}$ it follows that $g_{m-(s-1)t}^1 = \gamma_t$, or $m = st$. Finally, since g_t^1 imposes two conditions on g_{n+t}^{r+2} it imposes two conditions on g_n^r. Q.E.D.

It is perhaps worth remarking that by Lemmas 5.2, 5.3, and 3.5, formula (3.3), we can prove Castelnuovo's Theorem 3.4 in case we have equality in Castelnuovo's inequality (3.1) and $q \neq 2$. For the situation of Lemma 5.3 is that where the model of W_p in P^r lies on a rational normal scroll whose rulings cut out the g_t^1.

We now consider plane models for curves lying on rational normal scrolls in P^r.

5.4 Lemma. *Suppose W_p admits a complete simple g_N^R and a g_T^1, both without fixed points so that g_T^1 imposes two conditions on g_N^R. Then*

$$2p \leqslant (T-1)(2N - 2 - T(R-1)). \tag{5.1}$$

Proof. If $x_1 + \cdots + x_T$ is a general divisor in g_T^1 with T distinct points then $g_N^R - x_1$ has $x_2 + \cdots + x_T$ as a $(T-1)$-fold point since g_T^1 imposes two conditions on g_N^R. Consequently, if D_{R-2} is a general divisor of $R - 2$ points in $R - 2$ distinct divisors of g_T^1 then $g_N^R - D_{R-2}$ $(= g_{N-R+2}^2)$ will have $R - 2$ distinct $(T-1)$-fold points $P^{(1)}, \ldots, P^{(R-2)}$. g_T^1 will also impose two conditions on g^2_{N-R+2} and so $g^2_{N-R+2} = g^1_T + Q$ where Q is a $(N - R + 2 - T)$-fold point. Thus the plane curve C_{N-R+2} of degree $N - R + 2$ determined by g^2_{N-R+2} has genus p given by $2p \leq (N - R + 1)(N - R) - 2d$ where

$$2d = (R - 2)(T - 1)(T - 2) + (N - R + 2 - T)(N - R + 1 - T).$$

$2d$ is the minimum contribution of the singularities $P^{(i)}$ and Q to the double points of C_{N-R+2} suitably counted. Formula (5.1) follows by algebra. Q.E.D.

We will be interested in situations where the singularities $P^{(i)}$ and Q are disjoint from one another and from the other singularities of g^R_N as divisors on W_p. We now give a sufficient condition for this to be true.

5.5 Lemma. *Suppose it is known that $g^R_N - g^1_T$ $(= g^{R-2}_{N-T})$ is simple in Lemma 5.4. Then the plane curve C_{N-R+2} can be chosen so that all the singularities $P^{(i)}$ and Q are mutually disjoint and are disjoint from the other singularities of g^R_N, the singularities being considered as divisors on W_p.*

Proof. Let D_{R-3} $(= x_1 + \cdots + x_{R-3})$ be a divisor of $R - 3$ points corresponding to $R - 3$ distinct divisors in g^1_T and so that $g^{R-2}_{N-T} - D_{R-3}$ $(= g^1_{N-R+3-T})$ is without fixed points and so that these $R - 3$ divisors in g^1_T are disjoint from any singularity of g^R_N. Now choose x_{R-2} so that $g^1_{N-R+3-T} - x_{R-2}$ $(= Q)$ is disjoint from the $R - 2$ divisors in g^1_T determined by $x_1 + \cdots + x_{R-2}$ $(= D_{R-2})$ and all these divisors are disjoint from the singularities of g^R_N. Then $Q = g^{R-2}_{N-T} - D_{R-2} = g^R_N - D_{R-2} - g^1_T = g^2_{N-R+2} - g^1_T$. Q has no points common to the divisors of g^1_T determined by D_{R-2}. Q.E.D.

It turns out that there is a simple criterion which insures that the hypothesis of Lemma 5.5 holds.

5.6 Lemma. *Suppose in Lemma 5.4 we have $N < (R - 1)T$ and $R \geq 4$. Then $g^R_N - g^1_T$ is simple.*

Proof. Suppose g^{R-2}_{N-T} is composite. Then there is a t-sheeted cover $W_p \to W_q$ and a complete $g^{R-2}_{(N-T-f)/t}$ on W_q which lifts to the nonfixed points of g^{R-2}_{N-T}. Since g^1_T imposes at most two conditions on g^{R-2}_{N-T} we see that there is a $g^1_{T/t}$ on W_q which lifts to g^1_T on W_p. Suppose $q \neq 0$. Then $T/t \geq 2$ and on W_q: $g^{R-2}_{(N-T-f)/t} + g^1_{T/t} = g^{R+\varepsilon}_{(N-f)/t}$. The lift of $g^{R+\varepsilon}_{(N-f)/t}$ is included in g^R_N (assumed complete), $\varepsilon = 0$, and so g^R_N is composite. This contradiction shows that $q = 0$, $T = t$, and $R - 2 = (N - T - f)/T$ or $(R - 1)T = N - f \leq N$. This is the final contradiction. Q.E.D.

Notice that the proof of Lemma 5.6 shows that if g^{R-2}_{N-T} is composite then g^1_T imposes one condition on it. It can be shown by example that the numerical criterion of Lemma 5.6 cannot be sharpened.

It should be remarked that while as divisors on W_p the singularities of g_N^R, the $P^{(i)}$, and Q can be chosen mutually disjoint in the context of Lemma 5.5, there is nothing in the proof that guarentees that they may not fall together on the plane curve C_{N-R+2}. However, if this happens the bound in formula (5.1) cannot be sharp.

On the plane model C_{N-R+2} of Lemma 5.4 we can locate the original g_N^R. It is the family of (rational) curves of degree $R-1$ with a $(R-2)$-fold point at Q and a simple point at each $P^{(i)}$, for

$$\begin{aligned} g_N^R &\equiv g_{N-R+2}^2 + D_{R-2} \\ &\equiv (R-1)\,g_{N-R+2}^2 - (R-2)\,g_{N-R+2}^2 + D_{R-2} \\ &\equiv (R-1)\,g_{N-R+2}^2 - (R-2)(Q + g_T^1) + D_{R-2} \\ &\equiv (R-1)\,g_{N-R+2}^2 - (R-2)Q - \big((R-2)\,g_T^1 - x_1 - \cdots - x_{R-2}\big) \\ &\equiv (R-1)\,g_{N-R+2}^2 - (R-2)Q - P^{(1)} - \cdots - P^{(R-2)}. \end{aligned}$$

6. Consequences of equality in Castelnuovo's theorem for one linear series. Now we assume that W_p admits a simple g_n^r where we have equality in Castelnuovo's inequality, (3.1). If $r \neq 5$ then by Theorem 3.4 we know that W_p admits a g_T^1 imposing two conditions on g_n^r. By Lemma 3.6 we know that $T = m + 1 + t$ where $t \geq 0$. We now apply Lemma 5.4 with $2p = m(m-1)(r-1) + 2m(q-1)$, $N = m(r-1) + q$, $R = r$, and $T = m + 1 + t$. It follows that

$$0 \leq t(2q - 1 - r) - (r-1)t^2 = F(t).$$

Since $2 \leq q \leq r$ it follows that the only possible values for t are 0 and 1, the latter occuring only if $q = r$. In both of these cases $F(t) = 0$. Applying the results of the last section to this case we get the following theorem.

6.1 THEOREM. *Suppose $2p = m(m-1)(r-1) + 2m(q-1)$ and $n = m(r-1) + q$ where $q = 2, 3, \ldots, r-1$, or r; $m \geq 2$, and $r \neq 5$. Suppose W_p admits a simple g_n^r. Then W_p admits a g_T^1 with $T = m+1$ or $m+2$, the latter case occuring only if $q = r$. In either case W_p admits a plane model C_{n-r+2} of degree $n - r + 2$ with $r-2$ singularities $P^{(i)}$ of multiplicity $T-1$ and one singularity Q of multiplicity $n - r + 2 - T$. Several of the $P^{(i)}$ may be in the first neighborhood of Q although each of the $r-1$ singularities contributes to the double points of C_{n-r+2} as if it were an ordinary singularity. g_n^r is cut out on C_{n-r+2} by the rational curves of degree $r-1$ with a $(r-2)$-fold point at Q and simple points at each $P^{(i)}$.*

To obtain a model of W_p in which all the singularities are distinct in P^2 we could apply an appropriate quadratic transformation to C_{n-r+2}. Another way is as follows. By Castelnuovo's inequality g_{n+T}^{r+2} $(= g_n^r + g_T^1)$ is complete.

Since $g_{n+T}^{r+2} - g_T^1$ is simple we may apply Lemma 5.5. Since we will have equality in formula (5.1) in this case, the $r + 1$ singularities of the corresponding plane curve C_{n+T-r} will all occur at different points of P^2, for each singularity must contribute to the double points as if they were ordinary singularities. If $g_n^r - g_T^1$ is composite the singularities of multiplicity $T - 1$ of C_{n+T-r} are collinear and all the singularities $P^{(1)}, \ldots, P^{(r-2)}$ of C_{n-r+2} are all in the first neighborhood of Q.

If the $r - 1$ singularities of C_{n-r+2} are in general position, that is, if no $(l + 1)(l + 2)/2$ of them lie on a curve of degree l, then we can simplify the model by successive quadratic transformations. First transform with Q and $P^{(1)}$ and $P^{(2)}$ as fundamental points to obtain a $C_{n-r+2-(T-2)}$ with an $(n - r - 2(T - 2))$-fold point $Q^{(1)}$ and $r - 4$ $(T - 1)$-fold points $P^{(3)}$, $P^{(4)}, \ldots, P^{(r-2)}$. Second transform $C_{n-r+2-(T-2)}$ with $Q^{(1)}$ and $P^{(3)}$ and $P^{(4)}$ as fundamental points to obtain $C_{n-r+2-2(T-2)}$ with a $(n - r - 3(T - 2))$-fold point $Q^{(2)}$ and $r - 6$ $(T - 1)$-fold points. Continue.

If r is even, $(r - 2)/2$ such transformations yield a curve of degree $n - r + 2 - (r - 2)(T - 2)/2$ with a single singularity $Q^{((r-2)/2)}$ of multiplicity $n - r - r(T - 2)/2$. g_n^r is now cut out by the rational curves of degree $r/2$ with a $(r - 2)/2$-fold point at $Q^{((r-2)/2)}$.

If r is odd, $(r - 3)/2$ such transformations yield a curve of degree $n - r + 2 - (r - 3)(T - 2)/2$ with two singularities: $Q^{((r-3)/2)}$ of multiplicity $n - r - (r - 1)(T - 2)/2$, and $P^{(r-2)}$ of multiplicity $T - 1$. g_n^r is cut out by rational curves of degree $(r + 1)/2$ with an $((r - 1)/2)$-fold point at $Q^{((r-3)/2)}$ and passing simply through $P^{(r-2)}$.

It is worth remarking that in this case of r odd these latter curves give examples of Riemann surfaces for which equality is attained in the following classical inequality: if W_p admits a g_t^1 and a $g_{t'}^1$ then $p \leqslant (t' - 1)(t - 1)$. Here $t = T$ and $t' = n - r + 1 - (r - 1)(T - 2)/2$. In fact, in this case of r odd $g_n^r = g_{n-r+1-(r-1)(T-2)/2}^1 + ((r - 1)/2) g_T^1$.

Since the curves derived in the last three paragraphs are easily constructed, the question of the existence of Riemann surfaces where we have equality in Castelunovo's inequality is settled.

Concerning the uniqueness of the plane models, it is clear that those derived in Theorem 6.1 are an $(r - 2)$-dimensional family parametrized by D_{r-2}. However, if the singularities are in general position and r is even, then the plane curve $C_{n-r+2-(r-2)(T-2)/2}$ is unique since the corresponding linear series is $g_n^r - ((r - 2)/2) g_T^1$.

The case where $r = 5$ and W_p does not admit a g_T^1 imposing two conditions on g_n^r occurs only when $q = 2$ or $q = 4$. In these cases $g_{4m+q}^5 = 2g_{2m+q/2}^2$ where $g_{2m+q/2}^2$ is nonsingular since $2p = 4m(m - 1) + 2m(q - 1)$.

Again suppose W_p admits a simple g_n^r with equality in Castelnuovo's inequality. We want to show that g_T^1 is unique if $r \neq 3$. If $T = m + 1$ and there were two g_T^1's then each would impose two conditions on g_n^r by Lemma 3.8. In the proof of Theorem 6.1 (via Lemma 5.4) each g_{m+1}^1 would yield a different Q for the plane model C_{n-r+2} which would yield too much to the double points of this curve for the given genus. If W_p admits a g_{m+2}^1 the uniqueness follows as above since any g_{m+2}^1 also imposes two conditions on g_n^r. For in this case $q = r$, $g_{(m+1)(q-2)}^{q-2}$ is simple, and so $g_{(m+1)(q-2)}^{q-2} + g_{m+2}^1 = g_n^r$ (Lemmas 3.5 and 5.1). Thus g_T^1 is unique for both cases of T.

The uniqueness of g_T^1 gives results concerning the possible automorphisms (conformal self-maps) of a W_p admitting a g_n^r where we have equality in Castelnuovo's theorem, $r \neq 5$. For any automorphism of W_p must permute the fibers of g_T^1. If A is the full group of automorphisms and N is the normal subgroup of A of automorphisms that map each fiber of g_T^1 into itself, then A/N is isomorphic to a finite group of the Riemann sphere. Thus the automorphism group of W_p is severely restricted in much the same manner as are the hyperelliptic automorphism groups.

Since $f(r, r; n, n) < f(r; n)$ (formula (1.2)) we see that, on such a W_p, g_n^r is unique. This allows us to use the classical technique, formula (2.2), for counting the dimension of the locus in Teichmüller space of such W_p's. An elementary computation shows that the dimension D satisfies

$$2D = m^2(r-1) + m(2q + r + 1) + (4q - 2r - 8)$$

or

$$D = p + m(r+1) + (2q - r - 4).$$

7. Some remarks on the case of several linear series. There is an obvious way in which a surface W_p can admit many simple g_n^r's; that is, W_p can admit a simple g_{n+1}^{r+1} and as x varies over the surface we have an infinite number of simple $g_{n+1}^{r+1} - x$. However, Castelnuovo's original inequality, formula (3.1), imposes a bound on p if this is to occur. If $n + 1 = m'r + q' + 1$ where $1 \leqslant q' \leqslant r$ then $p \leqslant m'(m'-1)r/2 + m'q'$. In fact, if we denote the right-hand side of formula (4.1) by $2f(r, n, s)$ then for fixed r and n, $f(r, n, s)$ is a strictly decreasing function of s until $f(r, n, s) = f(r+1, n+1, 1)$ at which point it becomes constant. (The right-hand side of formula (3.1) is $f(r, n, 1)$.) We shall omit a proof of this, which is an elementary calculation, and show directly that if $p > f(r+1, n+1, 1)$ then W_p admits only a finite number of g_n^r's which are simple.

7.1 THEOREM. *Let $n = m'r + q'$ where $q' = 1, 2, \ldots, r-1$ or r. If $q' < r$ suppose that W_p admits m' simple g_n^r's. If $q = r$ suppose that W_p admits $m' + 1$ simple g_n^r's. (If $r = 1$ assume that no two g_n^1's are compounded of the same involution.) Then*

$$p \leqslant m'(m'-1)r/2 + m'q' \quad (= f(r+1, n+1, 1)). \tag{7.1}$$

PROOF. Let $g_1, g_2, \ldots, g_{m'}$ be distinct simple g_n^r's on W_p. Since $(m'-1)r + (r-1) + 1 < n$, $g_1 + g_2 + \cdots + g_{m'} = g_{m'n}^{m'r + m'(m'-1)r/2+\varepsilon}$. Call this series G. Then the Clifford index of G is $m'(q'-r) - 2\varepsilon \leqslant 0$. If this Clifford index is negative then formula (7.1) follows. If $q' < r$ then the Clifford index is indeed negative. If $q' = r$ and the Clifford index is zero, then $\varepsilon = 0$ and G is the canonical series. If the same were true of $G' = g_1 + g_2 + \cdots + g_{m-1} + g_{m+1}$ we would have $g_{m'} \equiv g_{m'+1}$, a contradiction. Thus the Clifford index of G or G' must be negative. Q.E.D.

We shall end Part I of this paper with examples of plane curves where we have equality in Theorem 4.3 for s g_n^r's, $s \geqslant 2$, thus showing that the bound in formula (4.1) is sharp. Suppose $n = m(rs-1) + q$. We tabulate below plane curves all of which are of degree $n - r + 2$ with one singularity Q, $r-2$ singularities $P^{(i)}$, $s-1$ singularities $R^{(j)}$, and one singularity S. Thus each curve will have $r + s - 1$ singularities in all. The case (v, u) indicates that q corresponds to the case $k = s - v$, and W_p admits a g_T^1 where $T = ms - v + 1 + u$ where $u = 0$ or 1.

Case (v, u)	Multiplicity of Q	Multiplicity of $P^{(i)}$	Multiplicity of $R^{(j)}$	Multiplicity of S	Range of q
	$(n-r+2-T)$	$(T-1)$	(h)	$(T-h)$	
(0, 0)	$n-r-ms+1$	ms	m	$(s-1)m+1$	$2 \leqslant q \leqslant r$
(0, 0)	$n-r-ms+1$	ms	$m+1$	$(s-1)m$	$2 \leqslant q \leqslant r$
(0, 1)	$n-r-ms$	$ms+1$	$m+1$	$(s-1)m+1$	$q = r$
(1, 0)	$n-r-ms+2$	$ms-1$	m	$(s-1)m$	$-r+2 \leqslant q \leqslant 1$
(1, 1)	$n-r-ms+1$	ms	$m+1$	$(s-1)m$	$q = 1$
(1, 1)	$n-r-ms+1$	ms	m	$(s-1)m+1$	$q = 1$
$(v, 0)$	$n-r-ms+1+v$	$ms-v$	m	$(s-1)m+1-v$	$2 \leqslant q+vr \leqslant r+1$
$(v, 1)$	$n-r-ms+v$	$ms+1-v$	m	$(s-1)m+2-v$	$q+vr = r+1$

In each case one g_n^r, say g_s, is cut out by curves of degree $r-1$ with a $(r-2)$-fold point at Q and passing simply through each $P^{(i)}$. For $l < s$, g_l is cut out by curves of degree r with an $(r-1)$-fold point at Q, and passing simply through all the $P^{(i)}$ and $R^{(l)}$ and S.

Another interesting model is obtained by transforming C_{n-r+2} by a quadratic transformation with fundamental points at Q, S, and an arbitrary nonsingular point x. The resulting model $C_{n-r+1+h}$ (h = order $R^{(j)}$) has a transformed Q, $r-1$ $P^{(i)}$'s, s $r^{(j)}$'s, and no S. For $l = 1, 2, \ldots, s$, g_l is cut out by the curves of degree r with a $(r-1)$-fold point at Q, and passing simply through all the $P^{(i)}$'s and $R^{(l)}$.

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